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A Test of the Exponential Distribution for Stand Structure Definition in Uneven-aged Loblolly-Shortleaf Pine Stands

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SUMMARY

In this study, 588 before-cut and **381** after-cut diameter (dbh) distributions of uneven-aged loblolly-shortleaf pine stands were fitted to two different forms of the exponential probability density function. The left truncated and doubly truncated forms of the exponential were used. The fitted distributions were evaluated using the Kolgomorov-Smirnov one-sample test at the 0.05 significance level. The rejection rates for the singly truncated form were 88 and 89 percent, respectively, for before-cut and after-cut stands. The rejections ran 72 and 77 percent for the doubly truncated function. Neither function adequately describes the structure of the stands used in the study, but each may have utility in certain regulation applications.

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INTRODUCTION

An uneven-aged forest is composed of trees of widely differing diameters (dbh). Selection management of such a forest would manipulate the diameter distribution or stand structure so that periodic timber harvests can be sustained indefinitely. It is not surprising, therefore, that many attempts have been made to somehow describe the stand structure of uneven-aged forests.

The French forester de Liocourt observed that the number of trees in successive diameter classes in an uneven-aged forest tended to decrease in a smooth geometric progression, and that this progression was apparently stable through time (Meyer 1952). The ratio of the number of trees in any two adjacent diameter classes tended to be a constant. The ratio of the number of trees in a given diameter class to those in the next larger class has been termed "q" in the literature. A stand structure is balanced if this q tends to be constant. If the q and other stand attributes tend to remain the same over time, then the stand is stable and exhibits little or no change as time progresses.

Meyer and Stephenson (1943) analyzed the diameter distributions of virgin beech-birch-maple-hemlock forests in Pennsylvania using graphs of the logarithms of numbers of trees over diameter class. A straight line relationship indicated that the stands were balanced, and the structures were apparently stable.

A balanced uneven-aged stand structure can be represented by the equation

$$Y_i = ke^{-aX_i}, \ a > 0 , \qquad (1)$$

where Y_i is the number of trees in the ith diameter class, X_i is the midpoint value of the diameter class in inches, and k and a are constants. Meyer (1952) demonstrated how the constants could be determined graphically. The q is related to this function in the following manner,

$$q = e^{aw}. \qquad (2)$$

The w is the width of the diameter class. Leak (1963) showed how equation (1) could be fitted by least squares to determine the coefficients a and k, and later (Leak 1964) developed a procedure to describe stand structure for unbalanced (nonconstant q) uneven-aged stands. In this procedure, the q's for 4-inch diameter classes are calculated, and these values are regressed on diameter. The q values for these large classes exhibited a linear relationship for Leak's unbalanced stand data.

Leak (1965) discussed the exponential distribution—the probability form of equation (1)—in an expository paper. It is

$$\begin{aligned} f(x) &= re^{-rx}, x \geq 0 \text{ and } r > 0, \qquad (3)\\ &= 0, \text{ elsewhere.} \end{aligned}$$

There is a functional relationship $q = \exp(rw)$ between the parameter r and q, where w is as previously defined. He also described the left-truncated form of the exponential p.d.f. (probability density function). The truncated form can be applied in situations where no trees below a certain threshold diameter are recorded. Although the functional form of the exponential, equation (1), has been used (e.g., Schmelz and Lindsey 1965) to describe uneven-aged stands, the probability form apparently has not.

The Weibull p.d.f. has gained wide acceptance in even-aged applications. It is

$$\begin{aligned} h(x) &= (\tfrac{c}{b})[(x-a)/b]^{c-1}\exp\{-[(x-a)/b]^c\}, \\ &\quad x \geq a, \ b > 0, \ c > 0, \\ &= 0, \text{ elsewhere.} \end{aligned}$$

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General properties are described by Bailey and Dell (1973). There have been two recent applications of the Weibull to uneven-aged situations. Hyink and Moser (1979) used the three-parameter Weibull for describing stand structures of uneven-aged mixed hardwood stands, and Stiff[1] employed the left-truncated two-parameter Weibull for representing the diameter distributions of mixed-species Appalachian hardwoods. The Weibull distribution degenerates into an exponential p.d.f. when the shape (or "c") parameter equals one.

Although more general distributions, like the Weibull, are available, the exponential has certain advantages. Like the Weibull, its cumulative distribution function has a closed form solution. But, unlike the Weibull, its parameter can be determined from the first sample moment and does not require iterative techniques for parameter estimation. The exponential would obviously be simple to use in the operational management and regulation of uneven-aged forests. These potential advantages justify investigating the exponential for use in describing diameter distributions, even though more flexible ones exist. In this study, the exponential's capacity to describe the diameter distributions of uneven-aged loblolly-shortleaf pine stands managed under the selection system was investigated.

PROCEDURE

Data

The data are from a cutting cycle study (Reynolds 1959, 1969), a methods-of-cutting study (Grano 1954), and unpublished research at the Crossett Experimental Forest and surrounding area in southern Arkansas. Average site index for loblolly pine (*Pinus taeda* L.) is about 90 feet at base age 50 (U.S. Forest Service 1976). The information was obtained from periodic 100-percent cruises of plots and compartments ranging from 2.5 to about 40 acres in size. Records were kept of stand inventories, harvest cuts, salvage cuts, and thinnings by one-inch dbh classes for trees 3.6 inches and larger. Collection of most of these data started in 1937 and continued into the late 1960's on a periodic basis.

All species were tallied in the methods-of-cutting study, and all were initially inventoried in the cutting cycle study by nominal lo-acre subcompartments. In 1948, all hardwoods on the cutting cycle study were controlled by injection or girdling, and thereafter tallies were kept only for pine by 40-acre compartments. Nine of the compartments from the cutting cycle study were inventoried in 1979 to provide additional data; the methods-of-cutting study was also remeasured.

Subcompartment data were used in lieu of compartment data, when available, to augment the information base from the cutting cycle study. Stand tables before cut and after cut were analyzed separately for the pines, loblolly and shortleaf (*Pinus echinata* Mill.). Hardwoods were not included. If a harvest cut occurred within half of a growing season of the stand inventory, an after-cut stand table was calculated by subtracting the cut tally from the before-cut stand table. Depicting stand structure after cut is of interest in projecting changes in diameter distributions of harvested stands over time. The total number of before-cut stand tables was 588; after-cut stand tables, 381.

Analysis

Since the data were collected only for trees in the 4-inch diameter class and above, the left truncated form of the exponential was used. It is

$$f(x) = re^{-r(x-a)},\ x \geq a,\ a>0,\ r>0, \qquad (4)$$
$$= 0,\ \text{elsewhere.}$$

The value of the parameter "a" is known and fixed at 3.5 inches. The parameter r is

$$r = 1/(\mu - a), \qquad (5)$$

In this case, the variable x is tree diameter. The population mean of x is known, because 100-percent inventories were conducted on each subcompartment or compartment. Mean stand diameter was calculated for each diameter distribution, and r determined by equation (5).

The left truncated exponential is bounded below by the parameter a, but is unbounded above. Thus use of the function implies that there is a small, but real probability that a tree of infinitely large size can occur. To evaluate the impact of this phenomenon, the doubly truncated form of the exponential p.d.f. was also fitted to the stand table data. It is (Aroian 1965)

$$g(x) = \frac{re^{-rx}}{e^{-ar} - e^{-br}},\ a \leq x \leq b,\ 0 < a < b,\ r > 0, \qquad (6)$$
$$= 0,\ \text{elsewhere,}$$

where x is tree diameter. The parameter a is known and fixed at 3.5 inches, and b was set equal to the upper limit of the largest diameter class encountered. The b parameter varies by compartment and the date of the 100-percent tree tally. The parameter r is the root of the equation

$$1/r - \mu + \frac{(ae^{-ar} - be^{-br})}{(e^{-ar} - e^{-br})} = 0. \qquad (7)$$

A solution can be found by using Newton's method for finding roots or some alternative numerical procedure.

[1]Stiff, C. T. 1979. Modeling the growth dynamics of natural mixed-species Appalachian hardwood stands. Unpublished Ph. D. Dissertation, Virginia Polytechnic Institute and State University, Blacksburg, 206 p.

The doubly truncated exponential was fitted with a equal to 3.5 inches, b equal to the upper limit of the largest diameter class, and r equal to the root of equation (7). Equation (7) can be re-written to illustrate how the parameter r for the doubly truncated exponential differs from that of the singly truncated form in terms of μ and the other parameters. This version is

$$1/r = \mu \quad (ae^{-ar} - be^{-br})/(e^{-ar} - e^{-br}). \tag{8}$$

Recall that $1/r = \mu - a$ in the singly truncated exponential. In equation (8), the population mean is reduced by a term which is a function of al1 three parameters of the distribution.

Goodness-of-fit was evaluated by comparing the actual cumulative distribution to the fitted cumulative using the Kolgomorov-Smirnov one-sample test with a 0.05 significance level.

RESULTS

The summary of the tests may be seen in table 1. The results are conclusive. Neither the singly nor doubly truncated exponential p.d.f.'s adequately describe the loblolly-shortleaf pine stands used in the analysis. The rejection rate for the singly truncated form was 88 and 89 percent respectively for before-cut and after-cut stands. For the doubly truncated distribution, the rejection rates were 72 percent and 77 percent. The doubly truncated version did perform somewhat better than the singly truncated p.d.f.

Most of the rejections occurred because of lack of fit in the lower diameter classes. Though the exponential is a simple distribution, its simplicity is bought at the expense of curve flexibility. Even the addition of an upper bound, the b parameter, did not help much.

Table 1.— ***Kolgomorou-Smirnou test for fitting the exponential distribution to stand tables of uneuen-aged loblolly-shortleaf pine***

Stand table	Number of observations	Null hypothesis rejections[1]	
		Left truncated exponential	Doubly truncated exponential
Before cut	588	519	422
After cut	381	341	291

[1]5-percent level.

Figure 1 illustrates a typical diameter distribution and a fitted doubly truncated exponential p.d.f., and table 2 lists the observed stand table and the corresponding estimates for both forms of the exponential. The Kolgomorov-Smirnov test was significant at the 0.05 level for this particular stand, an indication of inade-

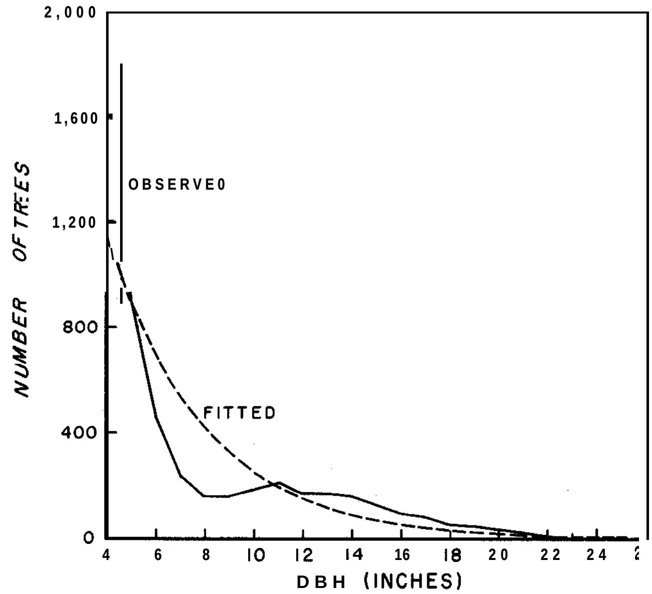

Figure 1.—*Observed and fitted doubly-truncated exponential diameter distribution, uneuen-aged loblolly-shortleafpine stand (39 acres).*

Table 2.-*Comparison of actual versus fitted diameter distributions, uneuen-aged loblolly-shortleaf pine stand (39 acres)*

Dbh (inches)	Observed	Left truncated exponential	Doubly truncated exponential
		Number of trees	
4	1840	1160	1152
5	893	896	893
6	457	693	691
7	237	535	535
8	156	414	415
9	159	320	321
10	184	247	249
11	209	191	193
12	169	148	149
13	168	114	116
14	158	88	90
15	126	68	69
16	90	53	54
17	80	41	42
18	53	31	32
19	49	24	25
20	33	19	19
21	26	15	15
22	8	11	12
23	4	9	9
24	1	7	7
25	1	5	5
26	0	4	4
27	0	3	3
28	3	2	3
29	1	2	2

quate fit to the data. The smaller diameters are much more subject to volatile change from such things as damage from ice storms and suppression mortality because of high stand densities. Also, these stands were essentially regulated by volume control in the sawtimber component and the smaller trees were not given strict objective regulation. The preponderance of trees in these classes together with their more erratic distributions make these sizes especially sensitive to goodness-of-fit tests. As seen in figure 1 and table 2, largest deviations from the actual distribution are in these lower classes. Furthermore, the Kolgomorov-Smirnov test is very rigorous in this application due to the large number of trees per observation. The number of trees for most observations in this study exceeded 1000 individuals. The critical value for the Kolgomorov-Smirnov test at the 5-percent level for samples exceeding 35 is $1.36/\sqrt{n}$ (Siegel 1956), where n is the sample size. For example with n = 100, the critical value is 0.136; when n = 1000, the critical value declines to .043.

In conclusion, the different forms of the exponential p.d.f. do not adequately describe the diameter distributions of uneven-aged loblolly-shortleaf pine stands used in this test. A more flexible curve form and, perhaps, a different density measure such as basal area are alternatives worthy of further investigation.

However, it should be pointed out that none of the stands in the study were regulated by consistent and objective application of a q to specify after-cut structures. Where such regulation is practiced, the exponential may have real value for specifying current residual structures and predicting future ones.

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Additional Keywords: Selection management, diameter classes.